SURFACES WITH ISOTHERMAL REPRESENTATION OF THEIR LINES OF CURVATURE AND THEIR TRANSFORMATIONS.*

(SECOND MEMOIR)

BY

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Introduction.

IN a former memoir with the same title † we established a transformation of surfaces with isothermal spherical representation of their lines of curvature into surfaces of the same kind, the transformation being such that lines of curvature on a surface and on a transform correspond, and the two surfaces constitute the envelope of a two-parameter family of spheres. This transformation was established by means of a theorem of MOUTARD, concerning partial differential equations of the LAPLACE type with equal invariants, ‡ and with the aid of a transformation of minimal surfaces discovered by THYBAUT. § In our discussion we did not take the equations of the transformation in the form given by THYBAUT, but in the form used by BIANCHI,|| in which case the parametric curves on the minimal surfaces are the asymptotic lines. Such a minimal surface and its THYBAUT transform are the focal sheets of a W-congruence, that is, a congruence upon whose focal sheets the asymptotic lines correspond.

The present paper deals with the same transformations obtained by a very different method as a result of which the analysis is much simpler. In § 1 it is shown that when the lines of a W-congruence are subjected to the LIE line-sphere transformation, the congruence of spheres envelope two surfaces upon which the lines of curvature correspond. If S and S_1 denote these two surfaces, and Σ and Σ_1 the two focal sheets of the original W-congruence, the transformation from Σ to Σ_1 carries with it a transformation from S to S_1 without quadrature. In § 1 is determined the characteristic property which Σ must have in order that the lines of curvature on S may have isothermal spherical representation, and the equations of the surfaces are given in simple form. In § 2 the determination of

* Presented to the Society, February 26, 1910.

† Transactions of the American Mathematical Society, vol. 9 (1908), pp. 149–177.

‡ BIANCHI, *Lezioni*, vol. 2, p. 69.

§ *Sur la déformation du paraboloide, etc.*; Annales de l'École Normale (3), vol. 14 (1897), pp. 65–69.

|| *Lezioni*, vol. 2, pp. 334–338.

surfaces Σ_1 and S_1 of the same kind as Σ and S respectively and standing in the above mentioned relation to them is carried through, and the problem is shown to be reducible to the integration of an ordinary differential equation of the first order and to quadratures.

When S is a minimal surface (cf. § 3), the transformation is that of THYBAUT and the equations take the very form which occurs in his memoir.*

In § 4 it is shown that the transformation of § 2 is identical with that established in the first memoir, and in § 5 we prove the existence of a "theorem of permutability" which shows that the knowledge of two transforms S_1 and S_2 of S carries with it the determination of a surface S' of the same kind, which is a transform of both S_1 and S_2; and, moreover, S' may be obtained without quadrature.

§ 1. *A Surface Σ and its Lie transform S.*

The coördinates, ξ, η, ζ, of a surface Σ, referred to its asymptotic lines, may be given by the Lelieuvre formulas †

$$\text{(1)} \qquad \frac{\partial \xi}{\partial u} = \begin{vmatrix} \nu_2 & \nu_3 \\ \frac{\partial \nu_2}{\partial u} & \frac{\partial \nu_3}{\partial u} \end{vmatrix}, \qquad \frac{\partial \xi}{\partial v} = -\begin{vmatrix} \nu_2 & \nu_3 \\ \frac{\partial \nu_2}{\partial v} & \frac{\partial \nu_3}{\partial v} \end{vmatrix},$$

and similar equations for η and ζ, obtained by permuting the quantities ν_1, ν_2, ν_3, which are solutions of an equation of the form

$$\text{(2)} \qquad \frac{\partial^2 \theta}{\partial u \partial v} = M\theta,$$

where in general M is a function of u and v. Moreover, ν_1, ν_2 and ν_3 are proportional to the direction-cosines of the normal to Σ.

If x, y, z denote the coördinates and X, Y, Z the direction-cosines of the normal to the surface S which is obtained from Σ by the line-sphere transformation of Lie, these quantities are given by ‡

$$\text{(3)} \qquad \begin{aligned} x + iy &= -\zeta - \xi\frac{\nu_1\xi + \nu_2\eta}{\nu_2 + \nu_3\xi}, & X + iY &= \frac{2\nu_2\xi}{\nu_2 + \nu_3\xi}, \\ x - iy &= \frac{\nu_1 - \nu_3\eta}{\nu_2 + \nu_3\xi}, & X - iY &= \frac{2\nu_3}{\nu_2 + \nu_3\xi}, \\ z &= \frac{\nu_1\xi + \nu_2\eta}{\nu_2 + \nu_3\xi}, & Z &= \frac{\nu_3\xi - \nu_2}{\nu_2 + \nu_3\xi}. \end{aligned}$$

From these equations we determine the following values for the fundamental

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* L. c.

† EISENHART, *Differential Geometry*, p. 193.

‡ DARBOUX, *Leçons*, vol. 1, p. 249.

quantities of S:

$$
(4) \qquad \begin{aligned} E &= \frac{\left(\frac{\partial \zeta}{\partial u} - \xi \frac{\partial \eta}{\partial u} + \eta \frac{\partial \xi}{\partial u}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \qquad F = 0, \\ G &= -\frac{\left(\frac{\partial \zeta}{\partial v} - \xi \frac{\partial \eta}{\partial v} + \eta \frac{\partial \xi}{\partial v}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \end{aligned}
$$

$$
(5) \qquad \begin{aligned} D &= 2 \frac{\partial \xi}{\partial u} \frac{\frac{\partial \zeta}{\partial u} - \xi \frac{\partial \eta}{\partial u} + \eta \frac{\partial \xi}{\partial u}}{(\nu_2 + \nu_3 \xi)^2}, \qquad D' = 0, \\ D'' &= -2 \frac{\partial \xi}{\partial v} \frac{\frac{\partial \zeta}{\partial v} - \xi \frac{\partial \eta}{\partial v} + \eta \frac{\partial \xi}{\partial v}}{(\nu_2 + \nu_3 \xi)^2}, \end{aligned}
$$

$$
(6) \qquad \mathcal{E} = \frac{4\left(\frac{\partial \xi}{\partial u}\right)^2}{(\nu_2 + \nu_3 \xi)^2}, \qquad \mathcal{F} = 0, \qquad \mathcal{G} = \frac{-4\left(\frac{\partial \xi}{\partial v}\right)^2}{(\nu_2 + \nu_3 \xi)^2},
$$

where $\mathcal{E}$, $\mathcal{F}$, $\mathcal{G}$ are the coefficients of the linear element of the spherical representation of S. From these results we have incidentally the well-known fact that the lines of curvature on S are parametric.

However, we are concerned entirely with surfaces whose lines of curvature admit an isothermal spherical representation. From (6) it is seen that a necessary and sufficient condition for this is that*

$$
(7) \qquad \frac{\partial \xi}{\partial u} - i \frac{\partial \xi}{\partial v} = 0,
$$

or by means of (1),

$$
(8) \qquad \frac{\partial}{\partial u} \log \frac{\nu_3}{\nu_2} + i \frac{\partial}{\partial v} \log \frac{\nu_3}{\nu_2} = 0.
$$

It has been assumed that the parameters are isometric, which assumption does not affect the generality of our solution.

If we put

$$
(9) \qquad u + iv = \alpha, \qquad u - iv = \beta,
$$

equation (8) gives the result

$$
(10) \qquad \nu_3 = \nu_2 A,
$$

where A denotes an arbitrary function of α. Since ν_2 and ν_3 satisfy (2), which

* The case where (7) is replaced by $\partial \xi / \partial u + i \cdot \partial \xi / \partial v = 0$ leads to similar results.

by the transformation (9) becomes

(11) $$\frac{\partial^2 \theta}{\partial \alpha^2} - \frac{\partial^2 \theta}{\partial \beta^2} = - iM\theta,$$

we must have

$$2A' \frac{\partial \nu_2}{\partial \alpha} + \nu_2 A'' = 0,$$

where the accents indicate differentiation with respect to the argument. From this equation we find

$$\nu_2 = \frac{\sqrt{B'}}{\sqrt{A'}},$$

where B denotes an arbitrary function of β, and where the accent denotes differentiation with respect to β. In consequence of (10), we have

(12) $$\nu_1 = \frac{\sqrt{B'}}{\sqrt{A'}}\sigma, \qquad \nu_2 = \frac{\sqrt{B'}}{\sqrt{A'}}, \qquad \nu_3 = \frac{\sqrt{B'}}{\sqrt{A'}}A,$$

where in general σ is a function of α and β, being a solution of

(13) $$\frac{\partial}{\partial \beta}\left[\frac{B'}{A'}\frac{\partial \sigma}{\partial \beta}\right] = \frac{\partial}{\partial \alpha}\left[\frac{B'}{A'}\frac{\partial \sigma}{\partial \alpha}\right],$$

which is obtained by expressing the fact that the functions (12) are solutions of equation (11).

§ 2. *The Determination of* Σ_1 *and of* S_1.

If θ_1 is a particular solution of equation (11), the functions $\bar{\nu}_i$, defined by*

(14) $$\frac{\partial}{\partial \alpha}(\theta_1 \bar{\nu}_i) = - \begin{vmatrix} \theta_1 & \nu_i \\ \frac{\partial \theta_1}{\partial \beta} & \frac{\partial \nu_i}{\partial \beta} \end{vmatrix}, \qquad \frac{\partial}{\partial \beta}(\theta_1 \bar{\nu}_i) = - \begin{vmatrix} \theta_1 & \nu_i \\ \frac{\partial \theta_1}{\partial \alpha} & \frac{\partial \nu_i}{\partial \alpha} \end{vmatrix},$$

determine a surface Σ_1, upon which the curves $u = \text{const.}$, $v = \text{const.}$, where u and v are given by (9), are the asymptotic lines; and Σ and Σ_1 are the focal surfaces of a W-congruence, formed by the lines joining corresponding points. Moreover, the coördinates, ξ_1, η_1, ζ_1, of Σ_1 are given by

(15) $$\xi_1 - \xi = \nu_2\bar{\nu}_3 - \nu_3\bar{\nu}_2, \quad \eta_1 - \eta = \nu_3\bar{\nu}_1 - \nu_1\bar{\nu}_3, \quad \zeta_1 - \zeta = \nu_1\bar{\nu}_2 - \nu_2\bar{\nu}_1.$$

In order that Σ_1 may be a surface of the same kind as Σ of § 1, we must have

(16) $$\bar{\nu}_1 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}\sigma_1, \quad \bar{\nu}_2 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}, \quad \bar{\nu}_3 = \frac{\sqrt{B_1'}}{\sqrt{A_1'}}A_1,$$

where A_1 and B_1 are functions of α and β alone to be determined and where, in general, σ_1 is a function of both α and β.

* EISENHART, *Differential Geometry*, pp. 417–419.

When the values for $\bar{\nu}_2$ and $\bar{\nu}_3$ are substituted in (14), we obtain four equations which are equivalent to

$$(17)\qquad \begin{aligned} \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} &= \frac{(\sqrt{A_1'})'}{\sqrt{A_1'}} + \frac{A_1'}{A - A_1}, \\ \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} &= \frac{(\sqrt{B'})'}{\sqrt{B'}} + \frac{\sqrt{B_1'}}{\sqrt{B'}}\frac{\sqrt{A'A_1'}}{A - A_1}, \end{aligned}$$

and

$$(18)\qquad \begin{aligned} \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} &= \sqrt{A'}\left(\frac{1}{\sqrt{A'}}\right)' + \frac{A'}{A - A_1}, \\ \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} &= -\frac{(\sqrt{B_1'})'}{\sqrt{B_1'}} + \frac{\sqrt{B'}}{\sqrt{B_1'}}\frac{\sqrt{A'A_1'}}{A - A_1}. \end{aligned}$$

The necessary and sufficient condition that these two values of $\partial\theta_1/\partial\alpha$ be equal is readily found to be

$$(19)\qquad \sqrt{A'A_1'} = \kappa_1(A - A_1),$$

where κ_1 denotes a constant.

In like manner the condition that the two values of $\partial\theta_1/\partial\beta$ be equivalent is reducible to

$$(20)\qquad \sqrt{B'B_1'} = \kappa_1(B - B_1).$$

In consequence of these results, equations (17) and (18) may be written

$$(21)\qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = \frac{(\sqrt{A_1'})'}{\sqrt{A_1'}} + \kappa_1\frac{\sqrt{A_1'}}{\sqrt{A'}}, \qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = \frac{(\sqrt{B'})'}{\sqrt{B'}} + \kappa_1\frac{\sqrt{B_1'}}{\sqrt{B'}},$$

and

$$(22)\qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\alpha} = -\frac{(\sqrt{A'})'}{\sqrt{A'}} + \kappa_1\frac{\sqrt{A'}}{\sqrt{A_1'}}, \qquad \frac{1}{\theta_1}\frac{\partial\theta_1}{\partial\beta} = -\frac{(\sqrt{B_1'})'}{\sqrt{B_1'}} + \kappa_1\frac{\sqrt{B'}}{\sqrt{B_1'}}.$$

When the expression (16) for $\bar{\nu}_1$ is substituted in (14), the resulting equations are reducible by means of (21) and (22) to

$$(23)\qquad \begin{aligned} &\frac{\sqrt{A'}}{\sqrt{A_1'}}\frac{\partial\sigma_1}{\partial\alpha} + \kappa_1(\sigma_1 - \sigma) + \frac{\sqrt{B'}}{\sqrt{B_1'}}\frac{\partial\sigma}{\partial\beta} = 0, \\ &\frac{\sqrt{B_1'}}{\sqrt{B'}}\frac{\partial\sigma_1}{\partial\beta} + \kappa_1(\sigma_1 - \sigma) + \frac{\sqrt{A_1'}}{\sqrt{A'}}\frac{\partial\sigma}{\partial\alpha} = 0. \end{aligned}$$

As these equations are linear, the determination of σ_1 requires quadratures only. Hence the problem of the determination of the surfaces Σ_1, when Σ is given in terms of its asymptotic lines, requires the solution of the ordinary differential equation (19) and quadratures.

When these funtions $\sigma_1, \xi_1, \eta_1, \zeta_1$ have been obtained from (23) and (15), we find the coördinates of a surface S_1 by means of equations analogous to (3). If x_1, y_1, z_1 denote the coördinates of S_1 and X_1, Y_1, Z_1 denote the direction cosines of the normal to S_1, we find that

$$
\begin{aligned}
&x_1 + iy_1 - (x+iy) = R\,[X_1 + iY_1 - (X+iY)],\\
(24)\qquad &x_1 - iy_1 - (x-iy) = R\,[X_1 - iY_1 - (X-iY)],\\
&z_1 - z = R(Z_1 - Z),
\end{aligned}
$$

where

$$
(25)\qquad R = \tfrac{1}{2}\,\frac{(A\sigma_1 - A_1\sigma)\xi + (A - A_1)\eta + \sigma_1 - \sigma}{A_1 - A}.
$$

Hence S and S_1 are the envelope of a two-parameter family of spheres whose radii are defined by (25). The coördinates, x_0, y_0, z_0, of the centers of these spheres are given by

$$
\begin{aligned}
&x_0 + iy_0 = x + iy - R(X + iY) = \xi\,\frac{\sigma_1 - \sigma}{A - A_1} - \zeta,\\
(26)\qquad &x_0 - iy_0 = x - iy - R(X - iY) = \frac{A_1\sigma - A\sigma_1}{A_1 - A},\\
&2z_0 = 2(z - RZ) = \frac{A_1\sigma - A\sigma_1}{A_1 - A}\,\xi + \eta + \frac{\sigma_1 - \sigma}{A_1 - A}.
\end{aligned}
$$

We shall refer to the above relation between S and S_1 as a transformation T_{κ_1}, thus indicating the constant κ_1 which appears in the formulas.

§ 3. *Transformation of Minimal Surfaces.*

Minimal surfaces are the only surfaces of negative curvature for which the following conditions are satisfied simultaneously

$$
E = G, \qquad \mathcal{E} = \mathcal{G}, \qquad F = 0, \qquad \mathcal{F} = 0.
$$

In consequence of (4) and (6), these conditions are equivalent to the following upon Σ:

$$
\left(\frac{\partial \xi}{\partial u}\right)^2 + \left(\frac{\partial \xi}{\partial v}\right)^2 = 0,
$$

$$
\left(\frac{\partial \zeta}{\partial u} - \xi\frac{\partial \eta}{\partial u} + \eta\frac{\partial \xi}{\partial u}\right)^2 + \left(\frac{\partial \zeta}{\partial v} - \xi\frac{\partial \eta}{\partial v} + \eta\frac{\partial \xi}{\partial v}\right)^2 = 0.
$$

If we replace the first of these equations by (7), we must replace the second by

$$
(27)\qquad \frac{\partial \zeta}{\partial \beta} - \xi\frac{\partial \eta}{\partial \beta} + \eta\frac{\partial \xi}{\partial \beta} = 0,
$$

if the functions D and D'', given by (5), are to differ in sign.

By means of (1) and (12) equation (27) is reducible to

$$\eta = \frac{1+AB}{A'}\frac{\partial\sigma}{\partial\alpha} - \sigma B. \tag{28}$$

If this equation be differentiated with respect to α and the result equated to the value of $\partial\eta/\partial\alpha$ obtainable from (1) and (12), the resulting equation is reducible by means of (13) to

$$\frac{\partial}{\partial\beta}\log\left(\frac{B'}{A'}\frac{\partial\sigma}{\partial\beta}\right) = \frac{B'A}{1+AB}.$$

From this we get, by integration,

$$\frac{B'}{A'}\frac{\partial\sigma}{\partial\beta} = f(\alpha)(1+AB),$$

where the function f remains to be determined. Hence, integrating again with respect to β, we find

$$\sigma = A'f\int\frac{1+AB}{B'}d\beta + f_1(\alpha). \tag{29}$$

When this value is substituted in (13), we obtain

$$fA - \left(\frac{f_1'}{A'}\right)' = \left[\frac{(A'f)'}{A'}\right]'\int\frac{d\beta}{B'} + \left\{\left[\frac{(A'f)'}{A'}\right]A + A'f\right\}'\int\frac{Bd\beta}{B'} \tag{30}$$

Differentiating with respect to β, we have

$$\left[\frac{(A'f)'}{A'}\right]' + B\left\{\left[\frac{(A'f)'}{A'}\right]A + A'f\right\}' = 0.$$

Hence the two terms of this equation must be zero, from which we get

$$A'f = c, \tag{31}$$

where c denotes a constant. When this value of f is substituted in (30), we obtain

$$f_1 = c\left[A\int\frac{Ad\alpha}{A'} - \int\frac{A^2d\alpha}{A'}\right] + c_1A + c_2, \tag{32}$$

where c_1 and c_2 denote constants.

From (13) it is seen that if σ is a solution of this equation, so also is $\sigma + c_1A + c_2$ where c_1 and c_2 are arbitrary constants. Hence for the present we shall put $c_1 = c_2 = 0$ and furthermore assume $c = 1$. We shall see presently that this choice of the constants does not restrict the generality of our solution. When these values are substituted in (29), we obtain

$$\sigma = \int\frac{1+AB}{B'}d\beta + A\int\frac{Ad\alpha}{A_1} - \int\frac{A^2d\alpha}{A'}. \tag{33}$$

When this value and (12) are substituted in (1), we find

$$\xi = B,$$

(34) $$\eta = \int \frac{A\,d\alpha}{A'} + B\int \frac{A^2\,d\alpha}{A'} - B\int \frac{d\beta}{B'} + \int \frac{B\,d\beta}{B'},$$

$$\zeta = -\int \frac{d\alpha}{A'} - B\int \frac{A\,d\alpha}{A'} - B\int \frac{B\,d\beta}{B'} + \int \frac{B^2\,d\beta}{B'}.$$

It is readily found that these values satisfy equations (27) and (28).

From (3) we obtain by means of (12) and (34)

$$x = \frac{1}{2}\left(\int \frac{1-A^2}{A'}\,d\alpha + \int \frac{1-B^2}{B'}\,d\beta\right),$$

(35) $$y = -\frac{i}{2}\left(\int \frac{1+A^2}{A'}\,d\alpha - \int \frac{1+B^2}{B'}\,d\beta\right),$$

$$z = \int \frac{A\,d\alpha}{A'} + \int \frac{B\,d\beta}{B'},^*$$

and

(36) $$X = \frac{A+B}{1+AB}, \qquad Y = \frac{i(A-B)}{1+AB}, \qquad Z = \frac{AB-1}{1+AB}.$$

From (35) we find that the linear element of S is

(37) $$ds^2 = \frac{(1+AB)^2}{A'B'}\,d\alpha\,d\beta,$$

and from (36) that the linear element of the spherical representation is

(38) $$ds'^2 = \frac{4A'B'}{(1+AB)^2}\,d\alpha\,d\beta.$$

The minimal surface $\bar{S}$ adjoint to S is given by equations of the form

$$\frac{\partial \bar{x}}{\partial \alpha} = i\frac{\partial x}{\partial \alpha}, \qquad \frac{\partial \bar{x}}{\partial \beta} = -i\frac{\partial x}{\partial \beta}.$$

Hence

$$\bar{x} = \frac{i}{2}\left(\int \frac{A^2-1}{A'}\,d\alpha - \int \frac{B^2-1}{B'}\,d\beta\right),$$

(39) $$\bar{y} = \frac{1}{2}\left(\int \frac{A^2+1}{A'}\,d\alpha + \int \frac{B^2+1}{B'}\,d\beta\right),$$

$$\bar{z} = i\int \frac{A\,d\alpha}{A'} - i\int \frac{B\,d\beta}{B'}.$$

* Since the equations of any minimal surface can be given this form, the particular choice of the constants in equation (32) did not limit the generality of the discussion.

If we substitute the value (33) for σ and also

$$\sigma_1 = \int \frac{1 + A_1 B_1}{B_1'} d\beta + A_1 \int \frac{A_1\, d\alpha}{A_1'} - \int \frac{A_1^2 d\alpha}{A_1'} \tag{40}$$

in equations (23), we find that these equations are identically satisfied in consequence of the relations (19) and (20). Hence the minimal surface S, defined by (35), and the minimal surface S_1, defined by equations of the same form as (35) but with A and B replaced by A_1 and B_1 respectively, constitute the envelope of a two-parameter family of spheres.

From (25) we find that the radius of the sphere is

$$R = \frac{(1 + AB)(1 + A_1 B_1)}{2\sqrt{A'B'} \cdot \sqrt{A_1' B_1'}}. \tag{41}$$

The coördinates of the center of the sphere are readily found by means of (26).

If $\bar{S}_1$ denotes the adjoint of S, it is defined by equations analogous to (39). From these equations we find that

$$\begin{aligned} \bar{x} - \bar{x}_1 &= \frac{i}{2\kappa_1^2}\left(\frac{AA_1 - 1}{A - A_1} - \frac{BB_1 - 1}{B - B_1}\right), \\ \bar{y} - \bar{y}_1 &= \frac{1}{2\kappa_1^2}\left(\frac{AA_1 + 1}{A - A_1} + \frac{BB_1 + 1}{B - B_1}\right), \\ \bar{z} - \bar{z}_1 &= \frac{i}{2\kappa_1^2}\left(\frac{A + A_1}{A - A_1} - \frac{B + B_1}{B - B_1}\right). \end{aligned} \tag{42}$$

If X_1, Y_1, Z_1 denote the direction-cosines of the parallel normals to S_1 and $\bar{S}_1$, they are given by equations similar to (36). From these values and (42) we find

$$\Sigma X(\bar{x} - \bar{x}_1) = 0, \qquad \Sigma X_1(\bar{x} - \bar{x}_1) = 0.$$

Hence $\bar{S}$ and $\bar{S}_1$ are the focal sheets of the congruence formed by the joins of corresponding points. Moreover, the curves

$$\alpha + \beta = \text{const.}, \qquad \alpha - \beta = \text{const.}$$

on $\bar{S}$ and $\bar{S}_1$ are the asymptotic lines. Hence this congruence is a W-congruence, and the transformation from $\bar{S}$ to $\bar{S}_1$ is the transformation discovered by Thybaut.*

§ 4. *Identification of the Transformation of* § 2 *with a Former Transformation.*

In this section we shall show that the transformation of § 2 is identical with one formerly established in a different manner.†

* Cf. Introduction.

† *Surfaces with isothermal representation, etc.*, these Transactions, vol. 9 (1908), pp. 149–177. A reference to an equation (a) on page b of this memoir will be indicated by $[(a)\ \text{p.}\ b]$.

Comparing (38) with [(2) p. 151], we have

$$e^{2\theta}=\frac{(1+AB)^2}{4A'B'}.$$

When an equation for S_1 analogous to (38) is compared with [(27) p. 156], we find that

$$\frac{\phi^2}{\omega^2}=\frac{(1+AB)^2(1+A_1B_1)^2}{16A'A_1'B'B_1'}. \tag{43}$$

This identification is permissible in view of the results of § 3 and § 1 of the present and former memoirs respectively.

If we compare (37) with [(1), p. 151], we see that the coördinates of the minimal surface in § 3 are twice as large as in the former memoir. Moreover, the radius of the sphere, as given by [(30′), p. 157], is ϕ/ω, and consequently one half of R as given by (41) and (43). Furthermore, in § 5 of the former memoir it was shown that the case of minimal surfaces in § 1 of that memoir is a particular case of the general transformation of § 4 of the same memoir. Hence, in order to identify the two transformations, it is necessary and sufficient to show that R as given by (25) is equivalent to [cf. (48), p. 161]

$$R=\frac{2}{\omega}\left[m\phi W_1+\left(\frac{\partial W}{\partial u}\frac{\partial \phi}{\partial u}-\frac{\partial W}{\partial v}\frac{\partial \phi}{\partial v}\right)-W(m\phi-\omega)\right], \tag{44}$$

where m is a constant and W, W_1 measure the distances from the origin to the tangent planes of a surface S and its transform S_1.

From (3) and similar equations for S_1 we have

$$W=-\frac{\eta+A\zeta}{1+AB},\qquad W_1=-\frac{\eta_1+A_1\zeta_1}{1+A_1B_1}.$$

By means of (1), (9) and (12) we find

$$\frac{\partial W}{\partial \alpha}=\frac{A'(B\eta-\zeta)}{(1+AB)^2},\qquad \frac{\partial W}{\partial \beta}=\frac{AB'(\eta+A\zeta)}{(1+AB)^2}+\frac{\sigma B'}{1+AB}.$$

From [(25), p. 156] and [(6), p. 151], we get (using the notation of the present memoir),

$$\Sigma X_1\frac{\partial X}{\partial \alpha}=-\frac{e^{-2\theta}}{m\phi}\frac{\partial \phi}{\partial \beta},\qquad \Sigma X_1\frac{\partial X}{\partial \beta}=-\frac{e^{-2\theta}}{m\phi}\frac{\partial \phi}{\partial \alpha},$$

which by means of (36) gives

$$\frac{\partial \phi}{\partial \alpha}=-\frac{m\phi\,[A(A_1B_1-1)+A_1-A^2B_1]}{2A'(1+A_1B_1)},$$

$$\frac{\partial \phi}{\partial \beta}=-\frac{m\phi\,[B(A_1B_1-1)+B_1-A_1B^2]}{2B'(1+A_1B_1)}.$$

When these values are substituted in (44), we get (25), provided that

$$m = \kappa_1^2.$$

Hence the two transformations are identical.

§ 5. *Theorem of Permutability.*

In this section we establish the following theorem of permutability:

*If a surface S with isothermal representation of its lines of curvature be transformed into two surfaces S_1 and S_2 of the same kind by transformations T_{κ_1} and T_{κ_2} respectively, there exists a surface S' which is the transform of S_1 and of S_2 by transformations T'_{κ_2} and T'_{κ_1} respectively; and this surface S' can be obtained without quadratures.**

In order that there may be such a transform S', there must exist functions $A^{(1)}$ and $B^{(1)}$ satisfying the following equations analogous to (19) and (20),

$$(45)\qquad \begin{aligned} \sqrt{A_1' A^{(1)\prime}} &= \kappa_1'(A_1 - A^{(1)}), & \sqrt{B_1' B^{(1)\prime}} &= \kappa_1'(B_1 - B^{(1)}), \\ \sqrt{A_2' A^{(1)\prime}} &= \kappa_2'(A_2 - A^{(1)}), & \sqrt{B_2' B^{(1)\prime}} &= \kappa_2'(B_2 - B^{(1)}), \end{aligned}$$

where for the present the constants κ_1' and κ_2' are undetermined. Also there must exist a function σ' satisfying equations analogous to (23), which by means of the latter can be given the form

$$(46)\qquad \begin{aligned} &\frac{\sqrt{A_1'}}{\sqrt{A^{(1)\prime}}}\frac{\partial \sigma'}{\partial \alpha} + \kappa_1'(\sigma' - \sigma_1) + \frac{\sqrt{B'}}{\sqrt{B^{(1)\prime}}}\left[\kappa_1(\sigma - \sigma_1) - \frac{\sqrt{A_1'}}{\sqrt{A'}}\frac{\partial \sigma}{\partial \alpha}\right] = 0, \\ &\frac{\sqrt{B^{(1)\prime}}}{\sqrt{B_1'}}\frac{\partial \sigma'}{\partial \beta} + \kappa_1'(\sigma' - \sigma_1) + \frac{\sqrt{A^{(1)\prime}}}{\sqrt{A'}}\left[\kappa_1(\sigma - \sigma_1) - \frac{\sqrt{B'}}{\sqrt{B_1'}}\frac{\partial \sigma}{\partial \beta}\right] = 0, \end{aligned}$$

and similar equations obtained by changing the subscripts 1 into 2.

Eliminating $\partial \sigma'/\partial \alpha$ and $\partial \sigma'/\partial \beta$ from these four equations, we obtain

$$(47)\qquad \begin{aligned} &(\kappa_1'\sqrt{B_1'} - \kappa_2'\sqrt{B_2'})\sigma' - (\kappa_1'\sqrt{B_1'}\sigma_1 - \kappa_2'\sqrt{B_2'}\sigma_2) \\ &\qquad + \frac{\sqrt{A^{(1)\prime}}}{\sqrt{A'}}\left[\kappa_1\sqrt{B_1'}(\sigma - \sigma_1) - \kappa_2\sqrt{B_2'}(\sigma - \sigma_2)\right] = 0, \\ &(\kappa_1'\sqrt{A_2'} - \kappa_2'\sqrt{A_1'})\sigma' - (\kappa_1'\sqrt{A_2'}\sigma_1 - \kappa_2'\sqrt{A_1'}\sigma_2) \\ &\qquad + \frac{\sqrt{B'}}{\sqrt{B^{(1)\prime}}}\left[\kappa_1\sqrt{A_2'}(\sigma - \sigma_1) - \kappa_2\sqrt{A_1'}(\sigma - \sigma_2)\right] = 0. \end{aligned}$$

* Cf. former memoir, l. c., p. 166.

By means of (19) and (20) equations (45) are reducible to

$$\sqrt{A^{(1)'}} = \frac{\kappa_1'\kappa_2'}{\kappa_1\kappa_2}\sqrt{A'}\,\frac{\kappa_1\sqrt{A_2'} - \kappa_2\sqrt{A_1'}}{\kappa_2'\sqrt{A_1'} - \kappa_1'\sqrt{A_2'}}, \tag{48}$$

and the same equation in the B's. When these values are substituted in (47), it is found that the two equations are consistent if and only if

$$\kappa_1' = \pm\kappa_1, \qquad \kappa_2' = \pm\kappa_2 \tag{49}$$

or

$$\kappa_1' = \pm\kappa_2, \qquad \kappa_2' = \pm\kappa_1, \tag{50}$$

where the same sign must be used in every case.

When the values (49) are used we get

$$\sqrt{A^{(1)'}} = \mp\sqrt{A'}, \qquad \sigma' = \sigma,$$

so that the case (49) is to be excluded.

For the values (50) equation (48) is reducible to

$$A^{(1)} = \frac{\kappa_1^2 A_2(A - A_1) - \kappa_2^2 A_1(A - A_2)}{\kappa_1^2(A - A_1) - \kappa_2^2(A - A_2)}, \tag{51}$$

and the two expressions (47) may be given the form

$$\begin{aligned}\sigma' = {} & \frac{\kappa_1^2(B - B_1) - \kappa_2^2(B - B_2)}{\kappa_1^2(A - A_1) - \kappa_2^2(A - A_2)}\,\frac{A_1 - A_2}{B_1 - B_2}\,\sigma \\ & + \frac{\kappa_1^2 - \kappa_2^2}{B_1 - B_2}\,\frac{\sigma_2(A - A_1)(B - B_2) - \sigma_1(A - A_2)(B - B_1)}{\kappa_1^2(A - A_1) - \kappa_2^2(A - A_2)}.\end{aligned} \tag{52}$$

Hence the theorem is established if these values satisfy (46), as they can be shown to do.

PRINCETON,
March 30, 1910.